Definitions and key facts for section 2.2

A $n \times n$ matrix A is said to be **invertible** if there is a $n \times n$ matrix C such that

$$CA = I \text{ and } AC = I.$$

In this case, C is a unique matrix which we call the **inverse** of A and denote by A^{-1}.

$$AA^{-1} = I = A^{-1}A$$

We call a matrix which is *not* invertible a **singular matrix**. Thus an invertible matrix is a **nonsingular matrix**.

Computing 2×2 **inverses**: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

We call the quantity $ad - bc$ the **determinant** of A and write

$$\det A = ad - bc.$$

Facts about inverse matrices Suppose A is an invertible $n \times n$ matrix with inverse A^{-1}, then we can conclude the following.

1. For each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
2. A^{-1} is also invertible and $(A^{-1})^{-1} = A$.
3. The transpose A^T is invertible, and the inverse of A^T is the transpose of A^{-1}. That is
$$(A^T)^{-1} = (A^{-1})^T.$$
4. If B is an invertible $n \times n$ matrix, then the product AB is invertible with
$$(AB)^{-1} = B^{-1}A^{-1}.$$

In fact, we can generalize fact 4: any product of invertible matrices is invertible, and the inverse is the product of the inverses in *reverse* order. For example:

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}.$$

Computing A^{-1} for larger matrices

Fact: If A is an invertible $n \times n$ matrix, then A is row equivalent to I_n and in this case, the reduced echelon form of $\begin{bmatrix} A & I_n \end{bmatrix}$ is $\begin{bmatrix} I_n & A^{-1} \end{bmatrix}$.